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# Summation theorems for multidimensional basic hypergeometric series by determinant evaluations

Michael Schlosser *,1

*Institut für Mathematik der Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria*

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**Abstract**

We derive summation formulas for a specific kind of multidimensional basic hypergeometric series associated to root systems of classical type. We proceed by combining the classical (one-dimensional) summation formulas with certain determinant evaluations. Our theorems include $A_r$ extensions of Ramanujan's bilateral ${}_1\psi_1$ sum, $C_r$ extensions of Bailey's very-well-poised ${}_6\psi_6$ summation, and a $C_r$ extension of Jackson's very-well-poised ${}_8\phi_7$ summation formula. We also derive multidimensional extensions, associated to the classical root systems of type $A_r$, $B_r$, $C_r$, and $D_r$, respectively, of Chu's bilateral transformation formula for basic hypergeometric series of Gasper–Karlsson–Minton type. Limiting cases of our various series identities include multidimensional generalizations of many of the most important summation theorems of the classical theory of basic hypergeometric series. 

*Keywords:* Basic hypergeometric; Determinants; $A_r$-series; $C_r$-series; Multidimensional summation theorems, associated to root systems

## 1. Introduction

The theory of basic hypergeometric series consists of many well-known summation and transformation theorems. In this paper, we derive multiple generalizations of many of the classical basic hypergeometric summation formulas. These extensions of a specific, natural, kind of multidimensional basic hypergeometric series are associated to root systems of classical type but are different from those studied by Milne et al. [10, 20–22]. The type of series appearing in this paper were first considered by Gustafson and Krattenthaler [11,12] who showed how to obtain multivariable summation and transformation formulas from determinant evaluations. Our theorems include $A_r$ extensions of Ramanujan's bilateral ${}_1\psi_1$ sum, $C_r$ extensions of Bailey's very-well-poised ${}_6\psi_6$

* Present address: Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210, USA.

*E-mail address:* mschloss@math.ohio-state.edu (M. Schlosser)

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summation, and a $C_r$ extension of Jackson's very-well-poised balanced ${}_8\phi_7$ summation formula. We also derive multidimensional extensions, associated to the classical root systems of type $A_r$, $B_r$, $C_r$, and $D_r$, respectively, of Chu's bilateral transformation formula for basic hypergeometric series of Gasper–Karlsson–Minton type. For explanations of the convention in naming the series as $A_r$, $B_r$, $C_r$, or $D_r$ series, the reader is referred to [3,26], or also the remark preceeding Lemma 6.3 in this paper. Limiting cases of our series identities include multidimensional generalizations of many of the most important summation theorems of the classical theory of basic hypergeometric series. As explicit examples we provide $C_r$ terminating and nonterminating ${}_6\phi_5$ summation formulas, and an $A_r$ extension of the $q$-Pfaff–Saalschütz summation. This research is part of the author's Ph.D. thesis [26, Ch. VII], written under the supervision of C. Krattenthaler.

Recently, Gustafson and Krattenthaler [11, Theorem 1.15] discovered an $A_r$ (or equivalently $U(r+1)$) extension of Ramanujan's bilateral ${}_1\psi_1$ sum. The work in their paper involved $A_r$ series of a new kind. In particular, Gustafson and Krattenthaler's $A_r$ ${}_1\psi_1$ sum is different from Milne's [20].

Before we review Ramanujan's ${}_1\psi_1$ sum and Gustafson and Krattenthaler's extension thereof we recall the standard definitions in basic hypergeometric series theory (cf. [8]). Let $q$ be a complex number such that $|q|<1$. Define

$$(a;q)_\infty := \prod_{j\geqslant 0}(1-aq^j)$$

and,

$$(a;q)_k := \frac{(a;q)_\infty}{(aq^k;q)_\infty} \tag{1.1}$$

$$= \prod_{j=0}^{k-1}(1-aq^j), \tag{1.2}$$

where equality (1.2) holds when $k$ is a non-negative integer. We also find it convenient to use the Gasper–Rahman notation

$$(a_1,\ldots,a_m;q)_k \equiv (a_1;q)_k\,(a_2;q)_k\cdots(a_m;q)_k$$

for simplifying our displays.

Ramanujan's classical ${}_1\psi_1$ summation formula (see [14] or [8, (5.2.1)]) reads:

**Theorem 1.1** (Ramanujan's classical ${}_1\psi_1$ sum). *Let $a$, $b$ and $z$ be indeterminate, and suppose that none of the denominators in* (1.3) *vanish. Then*

$$\sum_{k=-\infty}^{\infty}\frac{(a;q)_k}{(b;q)_k}z^k = \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \tag{1.3}$$

*provided the series terminates or* $|q|<1$ and $|b/a|<|z|<1$.

**Remark 1.2.** Ramanujan's bilateral ${}_1\psi_1$ summation formula is one of the most fundamental formulas of the theory of basic hypergeometric series. A simple and elegant proof was given by Ismail [15] who noted that Theorem 1.1 is an immediate consequence of the $q$-binomial theorem (which is the $b=q$ case of Theorem 1.1), and analytic continuation.

Gustafson and Krattenthaler's $A_r$ ${}_1\psi_1$ summation formula [11, Theorem 1.15] can be stated as follows. Here and in the following we use the notation $|\mathbf{k}|=k_1+k_2+\cdots+k_r$.

**Theorem 1.3** (Gustafson and Krattenthaler). *Let $x_1,\ldots,x_r$, $a$, $b$ and $z$ be indeterminate, let $r\geqslant 1$, and suppose that none of the denominators in* (1.4) *vanish. Then*

$$\sum_{k_1,\ldots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a;q)_{k_i}}{(b;q)_{k_i}}\cdot z^{|\mathbf{k}|}\,q^{\sum_{i=1}^{r}(i-r)k_i}\right)$$
$$=\prod_{i=1}^{r}\frac{(q,b/a,azq^{1-i},q^i/az;q)_\infty}{(b,q/a,zq^{1-i},bq^{i-1}/az;q)_\infty}, \tag{1.4}$$

*provided the series terminates or $|b/a|<|z|<|q|^{r-1}<1$.*

Gustafson and Krattenthaler proved Theorem 1.3 by a combination of the classical $r=1$ case (1.3) and the Vandermonde determinant evaluation.

One of the main purposes of this paper is to provide some more $A_r$ extensions of Ramanujan's ${}_1\psi_1$ summation formula, see Section 2. But we are also able to derive $C_r$ extensions of Bailey's bilateral very-well-poised ${}_6\psi_6$ summation, and Jackson's balanced very-well-poised ${}_8\phi_7$ summation formula, see Sections 3 and 4, respectively. Some important specializations of these summations are given in Section 5. Furthermore, in Section 6, we combine determinant evaluations with Chu's [5] remarkable bilateral transformation formula of Gasper–Karlsson–Minton type [7,18,23,8, Section 1.9] to deduce multiple versions of Chu's identity. It is suprising that we obtain identities associated to various root systems of classical type, and we may also employ different bases, $q_1,\ldots,q_r$, in our series (see Theorem 6.4). It is also possible to obtain other multiple extensions of Chu's transformation formula by using other determinants in our derivation.

Proceeding by essentially the same method as Gustafson and Krattenthaler in the proof of their $A_r$ ${}_1\psi_1$ summation theorem our derivations require certain determinant evaluations which are more general than the classical Vandermonde determinant evaluation. One of the determinant evaluations we utilize, Lemma A.1, comes from a determinant lemma [19, Lemma 34] (see Lemma A.2) which has been successfully used by Krattenthaler in the computation of generating functions for plane partitions and tableaux. This determinant lemma has also been used by Gessel and Krattenthaler [9] for deriving several $A_r$ basic hypergeometric series identities. Independently, a special case of Lemma A.1 was involved in [28] in the computation of biorthogonal rational functions.

The particular technique in this paper has already been used in [12] to provide new proofs and generalizations of the $A_r$ extensions of Heine's ${}_2\phi_1$ transformations which have been discovered in [11]. In fact, the reading of [12] was the starting point of the author's investigations for identities of this kind of series. We believe that the method of this paper, which is entirely elementary, will be useful for proving other multidimensional series identities as well.

## 2. ${}_1\psi_1$ Summation formulas

Our $A_r$ extensions of Ramanujan's ${}_1\psi_1$ summation formula (1.3), are the following.

**Theorem 2.1** ($A_r\,{}_1\psi_1$ summations). *Let* $x_1,\dots,x_r$, $a_1,\dots,a_r$, $b_1\dots,b_r$, $z_1,\dots,z_r,a$, $b$ *and* $z$ *be indeterminate, let* $r\geqslant 1$, *and suppose that none of the denominators in* (2.1), (2.2), *or* (2.3) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(b_i;q)_{k_i}}\cdot z^{|\mathbf{k}|}\,q^{\sum_{i=1}^{r}(i-r)\,k_i}\right)$$

$$=q^{-\binom{r}{2}}\prod_{1\leqslant i<j\leqslant r}\left(\frac{b_i/x_i-b_j/x_j}{1/x_i-1/x_j}\right)\prod_{i=1}^{r}\frac{(q,b_i/ax_i,ax_iz,q/ax_iz;q)_\infty}{(b_i,q/ax_i,zq^{1-i},b_i/ax_iz;q)_\infty},\tag{2.1}$$

*provided the series terminates or* $|b_i/ax_i|<|z|<|q|^{r-1}<1$ *for* $i=1,\dots,r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(bx_i;q)_{k_i}}\cdot z^{|\mathbf{k}|}\,q^{\sum_{i=1}^{r}(i-r)\,k_i}\right)$$

$$=\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-a_jx_i/a_ix_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,bx_i/a_i,a_izq^{1-i},q^i/a_iz;q)_\infty}{(bx_i,q/a_i,zq^{1-i},bx_i/a_iz;q)_\infty},\tag{2.2}$$

*provided the series terminates or* $|bx_i/a_i|<|z|<|q|^{r-1}<1$ *for* $i=1,\dots,r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(bx_i;q)_{k_i}}\,z_i^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)\,k_i}\right)$$

$$=\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-z_j/z_i}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,bq^{1-i}/a,ax_iz_iq^{1-i},q^i/ax_iz_i;q)_\infty}{(bx_i,q/ax_i,z_iq^{1-r},b/az_i;q)_\infty},\tag{2.3}$$

*provided the series terminates or* $|b/a|<|z_i|<|q|^{r-1}<1$ *for* $i=1,\dots,r$.

**Proof.** We start with the sum

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(b_i;q)_{k_i}}\,z_i^{k_i}\cdot q^{\sum_{i=1}^r (i-r)\,k_i}\right) \tag{2.4}$$

and specialize the parameters $a_i$, $b_i$, $z_i$ later. We have

$$q^{\sum_{i=1}^r (i-r)\,k_i}\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)=\prod_{1\leqslant i<j\leqslant r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\right)$$

$$=\prod_{1\leqslant i<j\leqslant r}(1/x_i-1/x_j)^{-1}\det_{1\leqslant s,t\leqslant r}\left(\left(q^{-k_s}/x_s\right)^{r-t}\right),$$

the last equation due to the Vandermonde determinant evaluation. Hence we may write (2.4), when multiplied by $\prod_{1\leqslant i<j\leqslant r}(1/x_i-1/x_j)$, as

$$\det_{1\leqslant s,t\leqslant r}\left(x_s^{t-r}\sum_{k_s=-\infty}^{\infty}\frac{(a_s;q)_{k_s}}{(b_s;q)_{k_s}}\left(z_s q^{t-r}\right)^{k_s}\right).$$

Now, to the sum inside the determinant we apply Ramanujan's classical ${}_1\psi_1$ summation (1.3), with $a\mapsto a_s$, $b\mapsto b_s$, and $z\mapsto z_s q^{t-r}$. Thus we obtain

$$\det_{1\leqslant s,t\leqslant r}\left(x_s^{t-r}\frac{(q,b_s/a_s,a_sz_sq^{t-r},q^{1+r-t}/a_sz_s;q)_\infty}{(b_s,q/a_s,z_sq^{t-r},b_sq^{r-t}/a_sz_s;q)_\infty}\right).$$

Now, by using linearity of the determinant with respect to rows, we take some factors out of the determinant and obtain

$$\prod_{i=1}^{r}\frac{(q,b_i/a_i,a_iz_i,q/a_iz_i;q)_\infty}{(b_i,q/a_i,z_i,b_i/a_iz_i;q)_\infty}\det_{1\leqslant s,t\leqslant r}\left(\left(\frac{x_s}{a_s}\right)^{t-r}\frac{(b_s/a_sz_s;q)_{r-t}}{(q/z_s;q)_{r-t}}\right). \tag{2.5}$$

The determinant in (2.5) cannot be evaluated in closed form in general. But we can choose different specializations of the parameters $a_s$, $b_s$, and $z_s$, for $s=1,\dots,r$, for which the determinant can be reduced to a product by means of Lemma A.1.

The simplest choice is $a_s\equiv a$, $b_s\equiv b$, and $z_s\equiv z$. In this case the determinant in (2.5) equals

$$a^{\binom{r}{2}}\prod_{i=1}^{r}\frac{(b/az;q)_{r-i}}{(q/z;q)_{r-i}}\det_{1\leqslant s,t\leqslant r}\left(x_s^{t-r}\right)$$

and the last determinant can be reduced to $\prod_{1\leqslant i<j\leqslant r}(1/x_i-1/x_j)$. Substituting these calculations and performing some other elementary manipulations, we easily recover (1.4).

By choosing different specializations of the parameters in (2.5) we will now prove cases (2.1), (2.3) of our theorem.

To prove (2.1), we set $a_s=ax_s$ and $z_s\equiv z$. In this case the determinant in (2.5) equals

$$a^{\binom{r}{2}}\prod_{i=1}^{r}(q/z;q)_{r-i}^{-1}\det_{1\leqslant s,t\leqslant r}\left((b_s/ax_sz;q)_{r-t}\right).$$

The determinant can be evaluated by means of Lemma A.1 with $X_s \mapsto b_s/x_s$, $A \mapsto 1/az$, $B \mapsto 0$, and $C \mapsto 0$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (2.1).

To prove (2.2), we set $b_s = bx_s$ and $z_s \equiv z$. In this case the determinant in (2.5) equals

$$\prod_{i=1}^{r}(q/z;q)_{r-i}^{-1} \det_{1\leqslant s,t\leqslant r}\left(\left(\frac{x_s}{a_s}\right)^{t-r}(bx_s/a_sz;q)_{r-t}\right). \tag{2.6}$$

The determinant can be evaluated by means of a limiting case of Lemma A.1. Namely, first multiply both sides of the $C \mapsto 0$ case of (A.1) with $(-B)^{\binom{r}{2}}q^{\binom{r}{3}}$ and then let $B \to \infty$ in the resulting identity to see that

$$\det_{1\leqslant s,t\leqslant r}(X_s^{t-r}(AX_s;q)_{r-t}) = \prod_{1\leqslant i<j\leqslant r}(1/X_i - 1/X_j). \tag{2.7}$$

Now evaluate the determinant in (2.6) by the $X_s \mapsto x_s/a_s$ and $A \mapsto b/z$ case of (2.7). Substituting our calculations and performing some elementary manipulations, we easily deduce (2.2).

To prove (2.3), we set $a_s = ax_s$ and $b_s \equiv bx_s$. In this case the determinant in (2.5) equals

$$a^{\binom{r}{2}} \det_{1\leqslant s,t\leqslant r}\left(\frac{(b/az_s;q)_{r-t}}{(q/z_s;q)_{r-t}}\right).$$

The determinant can be evaluated by means of Lemma A.1 with $X_s \mapsto 1/z_s$, $A \mapsto b/a$, $B \mapsto q$, and $C \mapsto 0$. Finally, substituting our calculations and performing some elementary manipulations, we arrive at (2.3). $\square$

**Remark 2.2.** There are two other cases where the determinant in (2.5) can be evaluated in closed form.

We may set $b_s = bx_s$ and $z_s \equiv zx_s/a_s$. In this case the determinant in (2.5) equals

$$\prod_{i=1}^{r}(b/z;q)_{r-i} \det_{1\leqslant s,t\leqslant r}\left(\left(\frac{a_s}{x_s}\right)^{r-t}(qa_s/x_sz;q)_{r-t}^{-1}\right). \tag{2.8}$$

The determinant can be evaluated by means of a limiting case of Lemma A.1. Namely, first multiply both sides of the $C \mapsto 0$ case of (A.1) with $(-A)^{-\binom{r}{2}}q^{-\binom{r}{3}}$ and then let $A \to \infty$ in the resulting identity to see that

$$\det_{1\leqslant s,t\leqslant r}(X_s^{r-t}(BX_s;q)_{r-t}^{-1}) = \prod_{i=1}^{r}(BX_i;q)_{r-1}^{-1} \prod_{1\leqslant i<j\leqslant r}(X_i - X_j). \tag{2.9}$$

Now evaluate the determinant in (2.8) by the $X_s \mapsto a_s/x_s$ and $B \mapsto q/z$ case of (2.9). Substituting our calculations and performing some other elementary manipulations leads to the following identity:

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(a_i;q)_{k_i}}{(bx_i;q)_{k_i}}\left(\frac{zx_i}{a_i}\right)^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)k_i}\right)$$
$$=\prod_{1\leqslant i<j\leqslant r}\left(\frac{x_i/a_i-x_j/a_j}{x_i-x_j}\right)\prod_{i=1}^{r}\frac{(q,bx_i/a_i,zx_iq^{1-r},q^r/zx_i;q)_\infty}{(bx_i,q/a_i,zx_iq^{1-r}/a_i,bq^{i-1}/z;q)_\infty},\tag{2.10}$$

provided the series terminates or $|bx_i/a_i|<|z|<|q|^{r-1}<1$ for $i=1,\dots,r$.

Actually, (2.10) is equivalent to (2.1), since(2.10) can be obtained by doing the replacements $k_i\mapsto -k_i$, $x_i\mapsto 1/x_i$, $b_i\mapsto q/a_i$, for $i=1,\dots,r$, $a\mapsto q/b$, and $z\mapsto q^{r-1}b/z$ in (2.1), and some elementary manipulations.

On the other hand, if we set $a_s=ax_s$ and $z_s\equiv zb_s/x_s$ in (2.5) the determinant equals

$$a^{\binom{r}{2}}\prod_{i=1}^{r}(1/az;q)_{r-i}\det_{1\leqslant s,t\leqslant r}((qx_s/b_sz;q)_{r-t}^{-1}).\tag{2.11}$$

The determinant can be evaluated by means of the $A,C\to 0$ limiting case of Lemma A.1, reading

$$\det_{1\leqslant s,t\leqslant r}((BX_s;q)_{r-t}^{-1})=B^{\binom{r}{2}}q^{2\binom{r}{3}}\prod_{i=1}^{r}(BX_i;q)_{r-1}^{-1}\prod_{1\leqslant i<j\leqslant r}(X_i-X_j).\tag{2.12}$$

Now evaluate the determinant in (2.11) by the $X_s\mapsto x_s/b_s$ and $B\mapsto q/z$ case of (2.12). Again, substituting our calculations and performing some other elementary manipulations we arrive at the following summation:

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i;q)_{k_i}}{(b_i;q)_{k_i}}\left(\frac{zb_i}{x_i}\right)^{k_i}\cdot q^{\sum_{i=1}^{r}(i-r)k_i}\right)$$
$$=\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-b_jx_i/b_ix_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(q,b_i/ax_i,azb_iq^{1-i},q^i/azb_i;q)_\infty}{(b_i,q/ax_i,zb_iq^{1-r}/x_i,q^{i-1}/az;q)_\infty},\tag{2.13}$$

provided the series terminates or $|b_i/ax_i|<|z|<|q|^{r-1}<1$ for $i=1,\dots,r$.

As in the above case we have not obtained a new identity, since (2.13) is equivalent to (2.2), where (2.13) can be obtained by doing the replacements $k_i\mapsto -k_i$, $x_i\mapsto 1/x_i$, $a_i\mapsto q/b_i$, for $i=1,\dots,r$, $b\mapsto q/a$, and $z\mapsto q^{r-1}a/z$ in (2.2), and some elementary manipulations.

**Remark 2.3.** The choice $b_i\equiv qa_i$ in (2.4) is a special case of the summation formula implied by Theorem 6.4, where we even could have started with different bases $q_i$ in the series (2.4). In this case most of the factors in (2.5) cancel.

## 3. ${}_6\psi_6$ Summation formulas

Before we state the $C_r$ extensions of Bailey's ${}_6\psi_6$ summation formula that we are going to prove, it might be convenient to recall Bailey's original ${}_6\psi_6$ summation. This is (cf. [2,8, (5.3.1)])

**Theorem 3.1** (Bailey's classical ${}_6\psi_6$ sum). *Let a, b, c, d and e be indeterminate, and suppose that none of the denominators in* (3.1) *vanish. Then*

$$\sum_{k=-\infty}^{\infty} \frac{1-aq^{2k}}{1-a} \frac{(b,c,d,e;q)_k}{(aq/b,aq/c,aq/d,aq/e;q)_k} \left(\frac{qa^2}{bcde}\right)^k$$
$$= \frac{(aq,aq/bc,aq/bd,aq/be,aq/cd,aq/ce,aq/de,q,q/a;q)_\infty}{(aq/b,aq/c,aq/d,aq/e,q/b,q/c,q/d,q/e,qa^2/bcde;q)_\infty}, \tag{3.1}$$

*provided the series terminates or* $|q|<1$ and $|qa^2/bcde|<1$.

**Remark 3.2.** Andrews [1] discusses some applications of Bailey's very-well-poised ${}_6\psi_6$ summation formula to number theory.

Our $C_r$ extensions of Bailey's bilateral very-well poised ${}_6\psi_6$ summation formula are the following.

**Theorem 3.3** ($C_r$ ${}_6\psi_6$ summations). *Let* $x_1,\dots,x_r$, $e_1,\dots,e_r$, $a$, $b$, $c$, $d$ *and* $e$ *be indeterminate, let* $r\geqslant 1$, *and suppose that none of the denominators in* (3.2) *or* (3.3) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg( \prod_{1\leqslant i<j\leqslant r} \left( \frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i-1/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^r \left( \frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$
$$\times \prod_{i=1}^r \frac{(bx_i,cx_i,dx_i,e_ix_i;q)_{k_i}}{(ax_iq/b,ax_iq/c,ax_iq/d,ax_iq/e_i;q)_{k_i}} \left(\frac{a^2q}{bcde_i}\right)^{k_i} \Bigg)$$
$$= a^{\binom{r}{2}} \prod_{1\leqslant i<j\leqslant r} \left( \frac{1/e_i-1/e_j}{1/x_i-1/x_j} \frac{1}{1-ax_ix_j} \right) \prod_{i=1}^r \frac{(ax_i^2q, aq^{2-i}/bc, aq^{2-i}/bd;q)_\infty}{(ax_iq/b,ax_iq/c,ax_iq/d;q)_\infty}$$
$$\prod_{i=1}^r \frac{(aq/be_i, aq^{2-i}/cd, aq/ce_i, aq/de_i, q, q/ax_i^2;q)_\infty}{(ax_iq/e_i, q/bx_i, q/cx_i, q/dx_i, q/e_ix_i, a^2q^{2-r}/bcde_i;q)_\infty}, \tag{3.2}$$

*provided the series terminates or* $|q|<1$ *and* $|a^2q^{2-r}/bcde_i|<1$ *for* $i=1,\dots,r$,

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \Bigg( \prod_{1\leqslant i<j\leqslant r} \left( \frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i-1/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^r \left( \frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$
$$\times \prod_{i=1}^r \frac{(bx_i,cx_i,dx_i/e_i,e_ix_i;q)_{k_i}}{(ax_iq/b,ax_iq/c,ae_ix_iq/d,ax_iq/e_i;q)_{k_i}} \left(\frac{a^2q}{bcd}\right)^{k_i} \Bigg)$$

$$= a^{\binom{r}{2}} \prod_{1\leqslant i<j\leqslant r} \left( \frac{1/e_i - 1/e_j}{1/x_i - 1/x_j} \frac{1 - e_i e_j/d}{1 - a x_i x_j} \right) \prod_{i=1}^{r} \frac{(a x_i^2 q, a q^{2-i}/bc, a e_i q/bd; q)_\infty}{(a x_i q/b, a x_i q/c, a e_i x_i q/d; q)_\infty}$$

$$\prod_{i=1}^{r} \frac{(aq/be_i, ae_i q/cd, aq/ce_i, aq/d, q, q/ax_i^2; q)_\infty}{(ax_i q/e_i, q/bx_i, q/cx_i, e_i q/dx_i, q/e_i x_i, a^2 q^{2-i}/bcd; q)_\infty}, \tag{3.3}$$

*provided the series terminates or* $|q| < 1$ *and* $|a^2 q^{2-r}/bcd| < 1$.

**Proof.** We start with the sum

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty} \left( \prod_{1\leqslant i<j\leqslant r} \left( \frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \frac{1 - a x_i x_j q^{k_i+k_j}}{1 - a x_i x_j} \right) \prod_{i=1}^{r} \left( \frac{1 - a x_i^2 q^{2k_i}}{1 - a x_i^2} \right) \right.$$

$$\left. \prod_{i=1}^{r} \frac{(bx_i, c_i x_i, d_i x_i, e_i x_i; q)_{k_i}}{(ax_i q/b, ax_i q/c_i, ax_i q/d_i, ax_i q/e_i; q)_{k_i}} \left( \frac{a^2 q}{b c_i d_i e_i} \right)^{k_i} \right) \tag{3.4}$$

and specialize the parameters $c_i$, $d_i$ later. We have

$$\prod_{1\leqslant i<j\leqslant r} \left( \frac{q^{-k_i}/x_i - q^{-k_j}/x_j}{1/x_i - 1/x_j} \frac{1 - a x_i x_j q^{k_i+k_j}}{1 - a x_i x_j} \right) = \prod_{1\leqslant i<j\leqslant r} [(1/x_j - 1/x_i)(1 - a x_i x_j)]^{-1}$$

$$\times (a/b)^{\binom{r}{2}} q^{-\binom{r}{3}} \det_{1\leqslant s,t\leqslant r} \left( (bq^{-k_s}/ax_s; q)_{r-t} \, (bx_s q^{k_s}; q)_{r-t} \right),$$

due to the $X_s \mapsto q^{-k_s}/x_s$, $A \mapsto b/a$, $B \mapsto 0$, and $C \mapsto a$ case of Lemma A.1. Hence, using the elementary identities [8, (I.13) and (I.18)]

$$(bq^{-k_s}/ax_s; q)_{r-t} = \frac{(b/ax_s; q)_{r-t} \, (ax_s q/b; q)_{k_s}}{(ax_s q^{1-r+t}/b; q)_{k_s}} q^{(t-r)k_s}$$

and

$$(bx_s q^{k_s}; q)_{r-t} = \frac{(bx_s; q)_{r-t} \, (bx_s q^{r-t}; q)_{k_s}}{(bx_s; q)_{k_s}},$$

we may write (3.7), when multiplied by $\prod_{1\leqslant i<j\leqslant r} [(1/x_j - 1/x_i)(1 - a x_i x_j)]$, as

$$(a/b)^{\binom{r}{2}} q^{-\binom{r}{3}} \det_{1\leqslant s,t\leqslant r} \left( (b/ax_s; q)_{r-t} \, (bx_s; q)_{r-t} \sum_{k_s=-\infty}^{\infty} \frac{1 - a x_s^2 q^{2k_s}}{1 - a x_s^2} \right.$$

$$\left. \times \frac{(bx_s q^{r-t}, c_s x_s, d_s x_s, e_s x_s; q)_{k_s}}{(ax_s q^{1-r+t}/b, ax_s q/c_s, ax_s q/d_s, ax_s q/e_s; q)_{k_s}} \left( \frac{a^2 q^{1+t-r}}{b c_s d_s e_s} \right)^{k_s} \right).$$

Now, to the sum inside the determinant we apply Bailey's classical ${}_6\psi_6$ summation, (3.2), with $a \mapsto ax_s^2$, $b \mapsto bx_s q^{r-t}$, $c \mapsto c_s x_s$, $d \mapsto d_s x_s$, and $e \mapsto e_s x_s$. Thus we obtain

$$(a/b)^{\binom{r}{2}} q^{-\binom{r}{3}} \det_{1\leqslant s,t\leqslant r} \Bigg( (b/ax_s;q)_{r-t}\,(bx_s;q)_{r-t} \frac{(ax_s^2 q, aq^{1-r+t}/bc_s;q)_\infty}{(ax_s q^{1-r+t}/b, ax_s q/c_s;q)_\infty}$$

$$\times \frac{(aq^{1-r+t}/bd_s, aq^{1-r+t}/be_s, aq/c_s d_s, aq/c_s e_s, aq/d_s e_s, q, q/ax_s^2;q)_\infty}{(ax_s q/d_s, ax_s q/e_s, q^{1-r+t}/bx_s, q/c_s x_s, q/d_s x_s, q/e_s x_s, a^2 q^{1-r+t}/bc_s d_s e_s;q)_\infty} \Bigg).$$

Now, by using linearity of the determinant with respect to rows, we take some factors out of the determinant and obtain

$$\prod_{i=1}^{r} \frac{(ax_i^2 q, aq/bc_i, aq/bd_i, aq/be_i, aq/c_i d_i, aq/c_i e_i, aq/d_i e_i, q, q/ax_i^2;q)_\infty}{(ax_i q/b, ax_i q/c_i, ax_i q/d_i, ax_i q/e_i, q/bx_i, q/c_i x_i, q/d_i x_i, q/e_i x_i, qa^2/bc_i d_i e_i;q)_\infty}$$

$$\times (a/b)^{\binom{r}{2}} q^{-\binom{r}{3}} \det_{1\leqslant s,t\leqslant r} \left( \frac{(bc_s/a;q)_{r-t}(bd_s/a;q)_{r-t}\,(be_s/a;q)_{r-t}}{(bc_s d_s e_s/a^2;q)_{r-t}} \right). \tag{3.5}$$

To evaluate the determinant in (3.5) we choose different specializations of the parameters $c_s$, $d_s$, for $s=1,\dots,r$, for which the determinant can be reduced to a product by means of Lemma A.1.

One choice is $c_s \equiv c$, and $d_s \equiv d$. In this case the determinant in (3.5) equals

$$\prod_{i=1}^{r} [(bc/a;q)_{r-i}(bd/a;q)_{r-i}] \det_{1\leqslant s,t\leqslant r} \left( \frac{(be_s/a;q)_{r-t}}{(bcde_s/a^2;q)_{r-t}} \right).$$

The determinant can be evaluated by means of Lemma A.1 with $X_s \mapsto e_s$, $A \mapsto b/a$, $B \mapsto bcd/a^2$, and $C \mapsto 0$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (3.2).

The other choice is $c_s \equiv c$, and $d_s \equiv d/e_s$. In this case the determinant in (3.5) equals

$$\prod_{i=1}^{r} \frac{(bc/a;q)_{r-i}}{(bcd/a^2;q)_{r-i}} \det_{1\leqslant s,t\leqslant r} ((bd/ae_s;q)_{r-t}(be_s/a;q)_{r-t}).$$

The determinant can be evaluated by means of Lemma A.1 with $X_s \mapsto e_s$, $A \mapsto b/a$, $B \mapsto 0$, and $C \mapsto d$. Finally, substituting our calculations and performing some other elementary manipulations leads to (3.3). □

**Remark 3.4.** Other multivariate extensions, associated to root systems, of the very-well-poised ${}_6\psi_6$ summation formula were derived by Gustafson [10] using difference equations. He used these higher-dimensional ${}_6\psi_6$ summations to obtain, by specialization and limits, the Macdonald identities for the affine root systems of classical type.

## 4. An ${}_8\phi_7$ summation formula

One of the most powerful results in the theory and application of classical one-dimensional basic hypergeometric series is Jackson's [17] summation formula for a terminating ${}_8\phi_7$ series, which is both balanced and very-well-poised.

**Theorem 4.1** (Jackson's classical ${}_8\phi_7$ summation). *Let $a, b, c$ and $d$ be indeterminate, let $n$ be a nonnegative integer and suppose that none of the denominators in* (4.1) *vanish. Then*

$$\sum_{k=0}^{n} \frac{1-aq^{2k}}{1-a} \frac{(a,b,c,d,a^2q^{n+1}/bcd,q^{-n};q)_k}{(q,aq/b,aq/c,aq/d,bcdq^{-n}/a,aq^{n+1};q)_k} q^k$$
$$=\frac{(aq,aq/bc,aq/bd,aq/cd;q)_n}{(aq/bcd,aq/d,aq/c,aq/b;q)_n}. \tag{4.1}$$

Theorem 4.1 is Eq. (2.6.2) of [8], where we have chosen to do the replacement $e \to a^2q^{n+1}/bcd$ explicitly.

We state our $C_r$ extension of Jackson's balanced very-well-poised ${}_8\phi_7$ summation formula.

**Theorem 4.2** (dedicated to Tejasi[2] A $C_r$ Jackson's sum). *Let $x_1,\dots,x_r$, $a,b,c$ and $d$ be indeterminate, let $N$ be a nonnegative integer, let $r\geqslant 1$, and suppose that none of the denominators in* (4.2) *vanish. Then*

$$\sum_{k_1,\dots,k_r=0}^{N} \Bigg( \prod_{1\leqslant i<j\leqslant r} \left( \frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^{r} \left( \frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$
$$\times \prod_{i=1}^{r} \frac{(ax_i^2,bx_i,cx_i,dx_i,a^2x_iq^{2-r+N}/bcd,q^{-N};q)_{k_i}}{(q,ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-1-N}/a,ax_i^2q^{1+N};q)_{k_i}} \cdot q^{\sum_{i=1}^{r} i k_i} \Bigg)$$
$$= \prod_{1\leqslant i<j\leqslant r} \left( \frac{1-ax_ix_jq^N}{1-ax_ix_j} \right) \prod_{i=1}^{r} \frac{(ax_i^2q,aq^{2-i}/bc,aq^{2-i}/bd,aq^{2-i}/cd;q)_N}{(aq^{2-r}/bcdx_i,ax_iq/d,ax_iq/c,ax_iq/b;q)_N}. \tag{4.2}$$

**Remark 4.3.** Note that all summation indices on the left-hand side of (4.2) have the same range. Summing over a cube is strange but essential for the determinant in our derivation of (4.2) to simplify. In fact, the series

$$\sum_{\substack{0\leqslant k_i\leqslant N_i\\ i=1,2,\dots,r}} \Bigg( \prod_{1\leqslant i<j\leqslant r} \left( \frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j} \frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j} \right) \prod_{i=1}^{r} \left( \frac{1-ax_i^2q^{2k_i}}{1-ax_i^2} \right)$$
$$\times \prod_{i=1}^{r} \frac{(ax_i^2,bx_i,cx_i,dx_i,a^2x_iq^{2-r+N_i}/bcd,q^{-N_i};q)_{k_i}}{(q,ax_iq/b,ax_iq/c,ax_iq/d,bcdx_iq^{r-1-N_i}/a,ax_i^2q^{1+N_i};q)_{k_i}} \cdot q^{\sum_{i=1}^{r} i k_i} \Bigg)$$

[2] Tejasi is the daughter of Gaurav Bhatnagar. She was born on October 27, 1995, in Ohio.

does not factor, except for $N_1=N_2=\cdots=N_r$. To our knowledge, such a phenomenon has not occurred so far with terminating multiple series associated to root systems.

Unfortunately, we cannot use Theorem 4.2 for deriving a multiple ${}_{10}\phi_9$ transformation.

**Proof of Theorem 4.2.** We have

$$q^{\sum_{i=1}^{r}(i-r)\,k_i}\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)$$

$$=\prod_{1\leqslant i<j\leqslant r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)$$

$$=\prod_{1\leqslant i<j\leqslant r}[(1/x_j-1/x_i)(1-ax_ix_j)]^{-1}\prod_{i=1}^{r}\frac{(q^{2-r-k_i}/cx_i;q)_{r-1}\,(ax_iq^{2-r+k_i}/c;q)_{r-1}}{(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}}$$

$$\times(a/b)^{\binom{r}{2}}\,q^{-\binom{r}{3}}\det_{1\leqslant s,t\leqslant r}\left(\frac{(bq^{-k_s}/ax_s;q)_{r-t}\,(bx_sq^{k_s};q)_{r-t}}{(q^{2-r-k_s}/cx_s;q)_{r-t}\,(ax_sq^{2-r+k_s}/c;q)_{r-t}}\right),$$

due to the $X_s\mapsto q^{-k_s}/x_s$, $A\mapsto b/a$, $B\mapsto q^{2-r}/c$, and $C\mapsto a$ case of Lemma A.1. Hence, using some elementary identities from [8, Appendix I], we may write the left-hand side of (4.2), when multiplied by $\prod_{1\leqslant i<j\leqslant r}[(1/x_j-1/x_i)(1-ax_ix_j)]$, as

$$(a^2/bc^2)^{\binom{r}{2}}\,q^{-3\binom{r}{3}}\prod_{i=1}^{r}[(aq^{2-r}/bc;q)_{i-1}(bq^{2+r-2i}/c;q)_{i-1}]^{-1}$$

$$\times\det_{1\leqslant s,t\leqslant r}\left((b/ax_s,bx_s;q)_{r-t}(c/ax_s,cx_s;q)_{t-1}\sum_{k_s=0}^{N}\frac{1-ax_s^2q^{2k_s}}{1-ax_s^2}\right.$$

$$\left.\times\frac{(ax_s^2,bx_sq^{r-t},cx_sq^{t-1},dx_s,a^2x_sq^{2-r+N}/bcd,q^{-N};q)_{k_s}}{(q,ax_sq^{1-r+t}/b,ax_sq^{2-t}/c,ax_sq/d,bcdx_sq^{r-1-N}/a,ax_i^2q^{1+N};q)_{k_s}}q^{k_s}\right).$$

Now, to the sum inside the determinant we apply Jackson's classical ${}_8\phi_7$ summation, Theorem 4.1, with the replacements $a\mapsto ax_s^2$, $b\mapsto bx_sq^{r-t}$, $c\mapsto cx_sq^{t-1}$, $d\mapsto dx_s$, and $n\mapsto N$. Thus we obtain

$$(a^2/bc^2)^{\binom{r}{2}}\,q^{-3\binom{r}{3}}\prod_{i=1}^{r}[(aq^{2-r}/bc;q)_{i-1}(bq^{2+r-2i}/c;q)_{i-1}]^{-1}$$

$$\times\det_{1\leqslant s,t\leqslant r}\left((b/ax_s,bx_s;q)_{r-t}(c/ax_s,cx_s;q)_{t-1}\right.$$

$$\left.\times\frac{(ax_s^2q,aq^{2-r}/bc,aq^{1-r+t}/bd,aq^{2-t}/cd;q)_N}{(aq^{2-r}/bcdx_s,ax_sq/d,ax_sq^{2-t}/c,ax_sq^{1-r+t}/b;q)_N}\right).$$

Now, we take some factors out of the determinant and obtain

$$(a^3/bc^4)^{\binom{r}{2}}\, q^{3N\binom{r}{2}-7\binom{r}{3}} \prod_{i=1}^{r} \frac{(cq^{-N}/ax_i;q)_{r-1}\,(cx_i;q)_{r-1}}{(aq^{2-r}/bc;q)_{i-1}\,(bq^{2+r-2i}/c;q)_{i-1}}$$
$$\times \prod_{i=1}^{r} \frac{(ax_i^2q, aq^{2-r}/bc, aq^{1-r+i}/bd, aq^{2-i}/cd;q)_N}{(aq^{2-r}/bcdx_i, ax_iq/d, ax_iq/c, ax_iq/b;q)_N}$$
$$\times \det_{1\leqslant s,t\leqslant r}\left(\frac{(bq^{-N}/ax_s;q)_{r-t}(bx_s;q)_{r-t}}{(q^{2-r}/cx_s;q)_{r-t}\,(ax_sq^{2-r+N}/c;q)_{r-t}}\right). \tag{4.3}$$

The determinant can be evaluated by means of Lemma A.1 with $X_s \mapsto 1/x_s$, $A \mapsto bq^{-N}/a$, $B \mapsto q^{2-r}/c$, and $C \mapsto aq^N$. Subsequently, substituting our calculations and performing some other elementary manipulations leads to (4.2). □

## 5. Specializations

First, we state an important limiting case of Theorem 4.2, namely an $A_r$ extension of Jackson's [16] $q$-analog of the Pfaff–Saalschütz formula [8, (1.7.2)], a summation theorem for a terminating and balanced ${}_3\phi_2$ series. The $q=1$ case of this classical summation theorem, the ${}_3F_2$ summation theorem [8, (1.7.1)], was originally found by Pfaff [24], and was rediscovered by Saalschütz [25].

**Theorem 5.1** (An $A_r q$-Pfaff–Saalschütz sum). *Let $x_1,\ldots,x_r$, and $a,b$ and $c$ be indeterminate, let $N$ be a nonnegative integer, let $r\geqslant 1$, and suppose that none of the denominators in* (5.1) *vanish. Then,*

$$\sum_{k_1,\ldots,k_r=0}^{N}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{1-q^{k_i-k_j}x_i/x_j}{1-x_i/x_j}\right)\prod_{i=1}^{r}\frac{(ax_i,bx_i,q^{-N};q)_{k_i}}{(q,cx_i,abx_iq^{r-N}/c;q)_{k_i}}\cdot q^{\sum_{i=1}^{r} i\,k_i}\right)$$
$$=\prod_{i=1}^{r}\frac{(cq^{1-i}/a, cq^{1-i}/b;q)_N}{(cx_i, cq^{1-r}/abx_i;q)_N}. \tag{5.1}$$

**Proof.** The proof is just as in the classical one-dimensional case. Take the limit $a\to 0$ after replacing $d$ by $aq/d$ in (4.2). Finally, relabel $c\mapsto a$ and $d\mapsto c$ in the resulting identity to obtain (5.1). □

Other important specializations of our Theorems 3.3 and 4.2 are non-terminating and terminating $C_r$ ${}_6\phi_5$ summations.

**Theorem 5.2** ($C_r$ nonterminating ${}_6\phi_5$ summations). *Let $x_1,\ldots,x_r$, $a,b,c$ and $d$ be indeterminate, let $r\geqslant 1$, and suppose that none of the denominators in* (5.2) *or* (5.3) *vanish. Then*

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)\prod_{i=1}^{r}\left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(ax_i^2,bx_i,cx_i,dx_i;q)_{k_i}}{(q,ax_iq/b,ax_iq/c,ax_iq/d;q)_{k_i}}\left(\frac{aq}{bcdx_i}\right)^{k_i}\right)$$
$$=\prod_{1\leqslant i<j\leqslant r}(1-ax_ix_j)^{-1}\prod_{i=1}^{r}\frac{(ax_i^2q,aq^{2-i}/bc,aq^{2-i}/bd,aq^{2-i}/cd;q)_\infty}{(aq^{2-r}/bcdx_i,ax_iq/d,ax_iq/c,ax_iq/b;q)_\infty},\tag{5.2}$$

*provided the series terminates or* $|q|<1$ *and* $|aq^{2-r}/bcdx_i|<1$ *for* $i=1,\dots,r$.

$$\sum_{k_1,\dots,k_r=0}^{\infty}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)\prod_{i=1}^{r}\left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(ax_i^2,bx_i,cx_i,d;q)_{k_i}}{(q,ax_iq/b,ax_iq/c,ax_i^2q/d;q)_{k_i}}\left(\frac{aq}{bcd}\right)^{k_i}\right)$$
$$=\prod_{1\leqslant i<j\leqslant r}\frac{(1-ax_ix_j/d)}{(1-ax_ix_j)}\prod_{i=1}^{r}\frac{(ax_i^2q,aq^{2-i}/bc,ax_iq/bd,ax_iq/cd;q)_\infty}{(aq^{2-i}/bcd,ax_i^2q/d,ax_iq/c,ax_iq/b;q)_\infty},\tag{5.3}$$

*provided the series terminates or* $|q|<1$ *and* $|aq^{2-r}/bcd|<1$.

**Proof.** For (5.2), let $e_i\equiv ax_i$ in (3.2), or, equivalently, $N\to\infty$ in (4.2). For (5.3), let $e_i\equiv ax_i$ and $d\mapsto ad$ in (3.3). $\square$

**Theorem 5.3** (A $C_r$ terminating ${}_6\phi_5$ summation). *Let* $x_1,\dots,x_r$, $a,b$ *and* $c$ *be indeterminate, let* $N$ *be a nonnegative integer, let* $r\geqslant 1$, *and suppose that none of the denominators in* (5.4) *vanish. Then*

$$\sum_{k_1,\dots,k_r=0}^{N}\left(\prod_{1\leqslant i<j\leqslant r}\left(\frac{q^{-k_i}/x_i-q^{-k_j}/x_j}{1/x_i-1/x_j}\,\frac{1-ax_ix_jq^{k_i+k_j}}{1-ax_ix_j}\right)\prod_{i=1}^{r}\left(\frac{1-ax_i^2q^{2k_i}}{1-ax_i^2}\right)\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(ax_i^2,bx_i,cx_i,q^{-N};q)_{k_i}}{(q,ax_iq/b,ax_iq/c,ax_i^2q^{1+N};q)_{k_i}}\left(\frac{aq^{1+N}}{bc}\right)^{k_i}\right)$$
$$=\prod_{1\leqslant i<j\leqslant r}\frac{(1-ax_ix_jq^N)}{(1-ax_ix_j)}\prod_{i=1}^{r}\frac{(ax_i^2q,aq^{2-i}/bc;q)_N}{(ax_iq/b,ax_iq/c;q)_N},\tag{5.4}$$

*provided the series terminates or* $|q|<1$ *and* $|aq^{2+N-r}/bc|<1$.

**Proof.** Let $d\mapsto q^{-N}$ in (5.3), or, equivalently, $d\to\infty$ in (4.2). $\square$

If we specialize our above theorems further, we obtain $A_r$ extensions of various important classical summation theorems such as the $q$-Gauß, $q$-Chu-Vandermonde, and $q$-binomial theorem.

## 6. Transformation formulas of Gasper–Karlsson–Minton type

Finally, we give multiple extensions of Chu's [5, Eq. (15)] general formula transforming a bilateral ${}_{p+2}\psi_{p+2}$ series of Gasper–Karlsson–Minton type into a multiple of a unilateral (one-sided) ${}_{p+2}\phi_{p+1}$ series of Gasper–Karlsson–Minton type. (We say that a ${}_{p+2}\psi_{p+2}$ series is of Gasper–Karlsson–Minton type if there are $p$ upper parameters $a_1,\ldots,a_p$ and $p$ lower parameters $b_1,\ldots,b_p$ such that each $a_i$ differs from $b_i$ by a nonnegative integer power of $q$, i.e. $a_i = b_i q^{m_i}$, $m_i \geqslant 0$, [8, Section. 1.9].) Our observation is quite interesting, namely, not only that we may employ several bases $q_1,\ldots,q_r$ in the multiple series but our calculations can also be carried out using various 'Vandermonde determinants', corresponding to the associated root systems. Further multilateral transformations may be deduced by employing other determinants in our series. Our intention is merely to give an idea how such calculations work.

Chu's general bilateral transformation formula reads

**Theorem 6.1** (Chu). *Let $a, c, d, z$ and $b_1,\ldots,b_p$ be indeterminate, let $N$ be an integer and $m_1,\ldots,m_p$ nonnegative integers, and suppose that none of the denominators in (6.2) vanish. Then*

$$\sum_{k=-\infty}^{\infty} \frac{(a, z, b_1 q^{m_1}, b_2 q^{m_2},\ldots,b_p q^{m_p};q)_k}{(c, qdz, b_1, b_2,\ldots,b_p;q)_k} (q^{1-N}/a)^k$$

$$= z^N \frac{(qz/a, c/z, qd, q;q)_\infty}{(qdz, q/z, q/a, c;q)_\infty} \prod_{j=1}^{p} \frac{(b_j/z;q)_{m_j}}{(b_j;q)_{m_j}}$$

$$\times \sum_{k=0}^{\infty} \frac{(1/d, qz/c, qz/b_1, qz/b_2,\ldots,qz/b_p;q)_k}{(q, qz/a, q^{1-m_1}z/b_1, q^{1-m_2}z/b_2,\ldots,q^{1-m_p}z/b_p;q)_k} (cdq^{N-|\mathfrak{m}|})^k, \qquad (6.1)$$

*provided the series terminate or $|q|<1$ and $q/a| < |q^N| < |q^{|\mathfrak{m}|}/cd|$. Here, we have used the notation $|\mathfrak{m}| = m_1 + \cdots + m_p$.*

**Remark 6.2.** Note, that when $d=1$, the sum on the right-hand side of (6.1) reduces to a single term, and hence, (6.1) reduces to a summation formula. On the other hand, when $c=q$ we obtain a ${}_{p+2}\phi_{p+1}$ transformation (where the $N=|\mathfrak{m}|$ case was derived in [7]). Jim Haglund [13] has stumbled over the $c=q$ case of (6.1) via rook theory. He has also noticed that (6.1) can be obtained by specializing a general transformation formula for bilateral series, due to Slater [27], [8, (5.4.4)].

We give multiple generalizations of (6.1) associated to the root systems $A_r$, $B_r$, $C_r$, and $D_r$ of classical type (cf. [4]).

The determinant evaluations we use in these cases are listed in the following lemma. We remark that the evaluations are basically the Weyl denominator factorizations of type $A_r$, $B_r$, $C_r$, and $D_r$, respectively (cf. [6, Lemma 24.3]).

**Lemma 6.3.** *The following determinant evaluations hold*:

$$\det_{1\leqslant i,j\leqslant r}\left(x_i^{r-j}\right)=\prod_{1\leqslant i<j\leqslant r}(x_i-x_j), \tag{A}$$

$$\det_{1\leqslant i,j\leqslant r}\left(x_i^{r-j}-x_i^{r+j-1}\right)=\prod_{1\leqslant i<j\leqslant r}[(x_i-x_j)(1-x_ix_j)]\prod_{i=1}^{r}(1-x_i), \tag{B}$$

$$\det_{1\leqslant i,j\leqslant r}\left(x_i^{r-j}-x_i^{r+j}\right)=\prod_{1\leqslant i<j\leqslant r}[(x_i-x_j)(1-x_ix_j)]\prod_{i=1}^{r}(1-x_i^2), \tag{C}$$

$$\frac{1}{2}\cdot\det_{1\leqslant i,j\leqslant r}\left(x_i^{r-j}+x_i^{r+j-2}\right)=\prod_{1\leqslant i<j\leqslant r}[(x_i-x_j)(1-x_ix_j)]. \tag{D}$$

**Proof.** Identities (A)–(D) are readily proved by the standard argument that proves Vandermonde-type determinant evaluations. Namely, first it is verified that the determinant on the left-hand side vanishes whenever one of the factors of the right-hand side vanishes. Then one checks that both sides are polynomials in $x_1,\dots,x_r$ of the same total degree. Hence, the determinant equals a constant times the right-hand side. The constant is easily computed by comparing coefficients of a 'maximal' or 'minimal' monomial. □

Let $G$ denote $A_r$, $B_r$, $C_r$, or $D_r$. Besides, let $\Delta_G\,[x_1,\dots,x_r]$ denote the product side of the $G$-Vandermonde as displayed in Lemma 6.3. Now, we can state a general multilateral transformation formula for basic hypergeometric series associated to $G$.

**Theorem 6.4.** *Let $a_i,c_i,d_i,z_i$ and $b_{i1},\dots,b_{ip}$ be indeterminate, let $N_i$ be integers and $m_{i1},\dots,m_{ip}$ nonnegative integers, for $i=1,\dots,r$, and suppose that none of the denominators in* (6.5) *vanish. Then*

$$\sum_{k_1,\dots,k_r=-\infty}^{\infty}\left(\frac{\Delta_G\left[q_1^{-k_1}/x_1,\dots,q_r^{-k_r}/x_r\right]}{\Delta_G\,[1/x_1,\dots,1/x_r]}\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(a_i,z_i,b_{i1}q_i^{m_{i1}},\dots,b_{ip}q_i^{m_{ip}};q_i)_{k_i}}{(c_i,q_id_iz_i,b_{i1},\dots,b_{ip};q_i)_{k_i}}\,(q_i^{1-N_i}/a_i)^{k_i}\right)$$
$$=\prod_{i=1}^{r}z_i^{N_i}\,\frac{(q_iz_i/a_i,c_i/z_i,q_id_i,q_i;q_i)_\infty}{(q_id_iz_i,q_i/z_i,q_i/a_i,c_i;q_i)_\infty}\prod_{i=1}^{r}\prod_{j=1}^{p}\frac{(b_{ij}/z_i;q_i)_{m_{ij}}}{(b_{ij};q_i)_{m_{ij}}}$$
$$\times\frac{\Delta_G\,[z_1/x_1,\dots,z_r/x_r]}{\Delta_G\,[1/x_1,\dots,1/x_r]}\sum_{k_1,\dots,k_r=0}^{\infty}\left(\frac{\Delta_G[q_1^{k_1}z_1/x_1,\dots,q_r^{k_r}z_r/x_r]}{\Delta_G\,[z_1/x_1,\dots,z_r/x_r]}\right.$$
$$\left.\times\prod_{i=1}^{r}\frac{(1/d_i,q_iz_i/c_i,q_iz_i/b_{i1},\dots,q_iz_i/b_{ip};q_i)_{k_i}}{(q_i,q_iz_i/a_i,q_i^{1-m_{i1}}z_i/b_{i1},\dots,q_i^{1-m_{ip}}z_i/b_{ip};q_i)_{k_i}}\,(c_id_iq_i^{N_i-|m_i|})^{k_i}\right), \tag{6.2}$$

*provided the series terminate or converge* (*where we have used the notation $|m_i|=m_{i1}+\cdots+m_{ip}$, for $i=1,\dots,r$*).

**Sketch of proof.** We start with the left-hand side of (6.2). Using Lemma 6.4, we substitute the corresponding determinant for $\Delta_G\left[q_1^{-k_1}/x_1,\ldots,q_r^{-k_r}/x_r\right]$, and by using linearity of the determinant with respect to rows, we write the whole series as a single determinant. Now we transform the terms in the determinant by Theorem 6.1. In the resulting determinant, again by using linearity of the determinant with respect to rows, we take some factors out and are left with a determinant for which we can apply Lemma 6.3 again, obtaining the right-hand side of (6.2). It is an easy exercise to verify that for the respective determinants of Lemma 6.3 the described calculations indeed work out well and yield (6.2). □

**Remark 6.5.** It is also possible to extend (6.1) by using other determinants, instead of the respective $G$-Vandermonde determinants of Lemma 6.3, in the analysis of our above sketch of proof.

## Appendix A. A determinant lemma

Here we provide a determinant lemma which we needed for proving our theorems.

**Lemma A.1.** *Let $X_1,\ldots,X_r$, $A,B$, and $C$ be indeterminate. Then there holds*

$$\det_{1\leqslant s,t\leqslant r}\left(\frac{(AX_s;q)_{r-t}\,(AC/X_s;q)_{r-t}}{(BX_s;q)_{r-t}\,(BC/X_s;q)_{r-t}}\right)=\prod_{1\leqslant i<j\leqslant r}[(X_j-X_i)(1-C/X_iX_j)]$$
$$\times A^{\binom{r}{2}}q^{\binom{r}{3}}\prod_{i=1}^{r}\frac{(B/A;q)_{i-1}\,(ABCq^{2r-2i};q)_{i-1}}{(BX_i;q)_{r-1}\,(BC/X_i;q)_{r-1}}. \tag{A.1}$$

This determinant evaluation follows easily from a determinant lemma of Krattenthaler [19, Lemma 34], which we state here without proof.

Let the *degree* of a *Laurent polynomial* $\sum_{i=M}^{N} a_i x^i$, $M,N\in\mathbb{Z}$, $a_i\in\mathbb{R}$ and $a_N\neq 0$, be defined by $\deg p:=N$.

**Lemma A.2** (Krattenthaler). *Let $X_1,X_2,\ldots,X_r$, $A_2,A_3,\ldots,A_r$, and $C$ be indeterminates. If $p_0$, $p_1$, …, $p_{r-1}$ are Laurent polynomials with* $\deg p_j\leqslant j$ *and* $p_j(C/X)=p_j(X)$ *for $j=0,1,\ldots,r-1$, then*

$$\det_{1\leqslant s,t\leqslant r}((A_r+X_s)\cdots(A_{t+1}+X_s)(A_r+C/X_s)\cdots(A_{t+1}+C/X_s)\cdot p_{t-1}(X_s))$$
$$=\prod_{1\leqslant i<j\leqslant r}(X_i-X_j)(1-C/X_iX_j)\prod_{i=1}^{r}A_i^{i-1}\prod_{i=1}^{r}p_{i-1}(-A_i) \tag{A.2}$$

*with the convention that empty products* (*like* $(A_r+X_s)\cdots(A_{t+1}+X_s)$ *for* $t=r$) *are equal to* 1. (*The indeterminate $A_1$, which occurs at the right-hand side of* (A.2), *in fact is superfluous since it occurs in the argument of a constant polynomial.*)

**Proof of Lemma A.1.** By taking some factors out of the determinant, we write the left-hand side of (A.1) as

$$\begin{aligned} &A^{2\binom{r}{2}} q^{2\binom{r}{3}} \prod_{i=1}^{r} \left( (BX_i;q)_{r-1}^{-1} (BC/X_i;q)_{r-1}^{-1} \right) \\ &\quad \times \det_{1\leqslant s,t\leqslant r} ((1/A - X_s)\cdots(q^{1+t-r}/A - X_s)(1/A - C/X_s)\cdots(q^{1-r+t}/A - C/X_s) \\ &\quad \times (BX_s q^{r-t};q)_{t-1}\,(BCq^{r-t}/X_s;q)_{t-1}). \end{aligned} \tag{A.3}$$

Now we apply Lemma A.2 with the replacements $X_s \mapsto -X_s$, $A_t \mapsto q^{t-r}/A$, $C \mapsto C$, and $p_j(X) \mapsto (-BXq^{r-1-j};q)_j \cdot (-BCq^{r-1-j}/X;q)_j$. A few simplifications give the product side of (A.1) $\square$

**Remark A.3.** The special case $X_s \equiv q^s$ of Lemma A.1 is equivalent to a determinant evaluation which Wilson [28] utilized to compute biorthogonal rational functions.

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